Automorphisms of the Unit Disk

Let $\mathbb{D} = \{z : |z| < 1$. We want to describe all conformal maps from $\mathbb{D}$ onto $\mathbb{D}$. We will postpone doing this and instead describe all linear fractional transformations T from $\partial\mathbb{D}$ onto $\partial\mathbb{D}$ that take $\mathbb{D}$ into $\mathbb{D}$. A linear fractional transformation takes circles to circles, so T must take all points in $\mathbb{D}$ to points in $\mathbb{D}$ and all points z with $|z| > 1$ to points z with $|z| > 1$; or all points in $\mathbb{D}$ to $|z| > 1$ and points with $|z| > 1$ to points in $\mathbb{D}$. This is proved using the intermediate value theorem applied to $|T(z)|$ and the fact that $|T(z)| = 1$ exactly when $|z| = 1$.

Theorem 1. *The linear fractional transformations that map $|z| = 1$ to $|z| = 1$ and $\mathbb{D}$ to $\mathbb{D}$ can be described by*

$$\lambda \frac{z - a}{1 - \bar{a}z}, \ |a| < 1, \ |\lambda| = 1;$$

and also by

$$\frac{az + \bar{b}}{bz + \bar{a}}, \ |a|^2 - |b|^2 = 1.$$

Proof. We'll organize the proof in steps. Assume (new meaning of the letters a, b, c, d).

$$Tz = \frac{az + b}{cz + d}.$$

First we prove $d \neq 0$. If $d = 0$, the condition $ad - bc \neq 0$ implies $bc \neq 0$. Hence T can be written as $\frac{a}{c} + \frac{b}{cz}$. This implies that $T(0) = \infty$, which can't happen. So $d \neq 0$.

Now we know $d \neq 0$. Next we consider $c = 0$. Then since $ad - bc \neq 0$, $a \neq 0$. We have

$$Tz = \frac{az + b}{d} = \frac{a}{d}(z + \frac{b}{a}).$$

The image of $|z| = 1$ by this T is a circle with center $\frac{b}{d}$ and radius $|\frac{a}{d}|$. This is supposed to be the circle $|z| = 1$, so $b = 0$ and $|a| = |d|$. This implies that $Tz = \lambda z$, with $|\lambda| = 1$. That is one of our cases.

Next we consider $d \neq 0, c \neq 0$, and prove that in this case $a \neq 0$. If $a = 0$, we have

$$Tz = \frac{b}{cz + d}.$$

This implies that $T(\infty) = 0$ and that is not possible.

Finally, we prove that $b \neq 0$ when $adc \neq 0$. If $b = 0$, then

$$Tz = \frac{az}{cz + d}.$$

Then

$$|az|^2 = |cz|^2 + |d|^2 + 2Re(c\bar{d}z),$$
$$|a|^2 = |c|^2 + |d|^2 + 2Re(c\bar{d}z).$$

Let $c\bar{d} = re^{it}$ and $z = e^{i\theta}$. Then $2Re(c\bar{d}z) = re^{i(t+\theta)}$ and this varies with θ unless $r = 0$. This implies $c\bar{d} = 0$, which is contrary to our assumption. So $b \neq 0$.

Now introduce new letters and write T as

$$Tz = \lambda \frac{z-a}{1-dz}.$$

Since $Ta = 0$, $|a| < 1$. Also $T(0) = -\lambda a$ so $|\lambda a| < 1$. $\square$

The following relations, when $|z| = 1$,

$$|\lambda z|^2 + |a\lambda|^2 - 2Re(\bar{a}|\lambda|^2 z) = 1 + |dz|^2 + 2Re(dz),$$
$$|\lambda|^2 + |a|^2||\lambda|^2 = 1 + |d|^2 + 2Re((\bar{a}|\lambda|^2 - d)z),$$

imply

$$d = |\lambda|^2 \bar{a},$$
$$|\lambda|^2 + |a|^2||\lambda|^2 = 1 + |d|^2,$$

by an argument similar to a previous argument. Substituting, we get a quadratic equation for $|\lambda|^2$,

$$|\lambda|^4 - (1 + |a|^2)|\lambda|^2 + 1,$$

with solutions $|\lambda|^2 = 1, \frac{1}{|a|^2}$. Since $|a\lambda| < 1$, the second solution is ruled out. So $|\lambda| = 1, d = \bar{2}$, and these are the only possible linear fractional transformations that map $\mathbb{D}$ onto $\mathbb{D}$. It's easy to verify that they do map $\mathbb{D}$ onto $\mathbb{D}$.

By rescaling by, we can produce the second form.